Linije krivine

Krivina K površi u tački (u,v) u pravcu (du, dv) se računa po formuli

$$K = \frac{F_2}{F_1} = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}$$

Pravci za koje K ima maksimalnu i minimalnu vrijednost u fiksiranoj tački (u,v) zovu se glavni pravci u toj tački.

Kriva na površi čija tangenta u svakoj tački ima pravac jednog od glavnih pravaca u toj tački, zove se linija krivine krive te površi.

Diferencijalna jednačina linije krivine je

$$d\vec{r} \cdot (\vec{n} \times d\vec{n}) = 0$$

ili što je ekvivalentno

$$\begin{vmatrix} dv^2 & -du\,dv & du^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0$$

(#) Odrediti linije krivine površi $\vec{r}=(u, v, u^2+v^2)$.

Rj. Diferencijalna jednačina linije krivine je

$$d\vec{r}\cdot(\vec{n}\times d\vec{n})=0$$

$$\vec{r}\,'_u=(1,0,2u)$$
$$\vec{r}\,'_v=(0,1,2v)$$

$$\vec{n}=\vec{r}\,'_u\times\vec{r}\,'_v=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & 2u\\ 0 & 1 & 2v\end{vmatrix}=(-2u,-2v,1)$$

$$d\vec{r}=(du,\ dv,\ 2u\,du+2v\,dv)$$

$$d\vec{n}=(-2du,\ -2dv,\ 0)$$

Iz $d\vec{r}\cdot(\vec{n}\times d\vec{n})=0$ imamo

$$\begin{vmatrix} du & dv & 2u\,du+2v\,dv\\ -2u & -2v & 1\\ -2du & -2dv & 0\end{vmatrix}=0$$

$$\overset{III\,v+I\,v\cdot 2}{\Longleftrightarrow}\begin{vmatrix} du & dv & 2u\,du+2v\,dv\\ -2u & -2v & 1\\ 0 & 0 & 4u\,du+4v\,dv\end{vmatrix}=0$$

$$(4u\,du+4v\,dv)\begin{vmatrix} du & dv\\ -2u & -2v\end{vmatrix}=0 \qquad /:8$$

$$(u\,du+v\,dv)(-v\,du+u\,dv)=0$$

$$-uv\,du^2+(u^2-v^2)\,du\,dv+uv\,dv^2=0 \qquad /:dv^2$$

$$-uv\left(\frac{du}{dv}\right)^2+(u^2-v^2)\frac{du}{dv}+uv=0$$

$$D=(u^2-v^2)^2+4u^2v^2=u^4-2u^2v^2+v^4+4u^2v^2$$
$$=u^4+2u^2v^2+v^4=(u^2+v^2)^2$$

$$\left(\frac{du}{dv}\right)_{1,2}=\frac{-(u^2-v^2)\pm(u^2+v^2)}{-2uv}$$

$$\left(\frac{du}{dv}\right)_1=\frac{-u^2+v^2-u^2-v^2}{-2uv}=\frac{-2u^2}{-2uv}=\frac{u}{v} \Rightarrow$$

$$\frac{du}{dv}=\frac{u}{v}$$
$$\frac{du}{u}=\frac{dv}{v}$$
$$\ln u=\ln v+\ln C$$
$$u=Cv$$

$$\left(\frac{du}{dv}\right)_2=\frac{-u^2+v^2+u^2+v^2}{-2uv}=\frac{2v^2}{-2uv}=-\frac{v}{u} \Rightarrow$$

$$\frac{du}{dv}=-\frac{v}{u}$$
$$u\,du=-v\,dv$$
$$u^2+v^2=C$$

Tražene linije krivine su

$$\vec{r}_1=(Cv,\ v,\ C^2v^2+v^2)$$
$$\vec{r}_2=(u,\ \pm\sqrt{C-u^2},\ C)$$

Odrediti linije krivine površi $\vec{r} = (u\cos v, u\sin v, av)$.

Rj. Za diferencijalnu jednačinu linije krivine možemo uzeti

$$d\vec{n}_0 = \lambda\, d\vec{r}.$$

$$\vec{r}_u' = (\cos v, \sin v, 0)$$

$$\vec{r}_v' = (-u\sin v, u\cos v, a)$$

$$\vec{n} = \vec{r}_u' \times \vec{r}_v' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & a \end{vmatrix} = (a\sin v, -a\cos v, u)$$

$$\vec{n}_0 = \frac{\vec{n}}{|\vec{n}|} = \frac{1}{\sqrt{a^2+u^2}}(a\sin v, -a\cos v, u)$$

$$d\vec{n}_0 = \frac{\partial \vec{n}_0}{\partial u}du + \frac{\partial \vec{n}_0}{\partial v}dv, \quad \frac{1}{\sqrt{a^2+u^2}} = (a^2+u^2)^{-\frac{1}{2}}, \quad \left(\frac{1}{\sqrt{a^2+u^2}}\right)' = -\frac{1}{2}\cdot(a^2+u^2)^{-\frac{3}{2}}\cdot 2u =$$

$$= \frac{-u}{\sqrt{(a^2+u^2)^3}}$$

$$\vec{n}_0 = \left(\frac{a\sin v}{\sqrt{a^2+u^2}}, \frac{-a\cos v}{\sqrt{a^2+u^2}}, \frac{u}{\sqrt{a^2+u^2}}\right)$$

$$\left(\frac{u}{\sqrt{a^2+u^2}}\right)' = \frac{\sqrt{a^2+u^2} - u\frac{\frac{1}{2}}{\sqrt{a^2+u^2}}\cdot 2u}{a^2+u^2} = \frac{\sqrt{a^2+u^2} - \frac{u^2}{\sqrt{a^2+u^2}}}{a^2+u^2} \begin{matrix} /\cdot\sqrt{a^2+u^2} \\ /\cdot\sqrt{a^2+u^2} \end{matrix}$$

$$= \frac{a^2+u^2-u^2}{(a^2+u^2)\sqrt{a^2+u^2}} = \frac{a^2}{\sqrt{(a^2+u^2)^3}}$$

Prema tome

$$\vec{n}_{0_u}' = \frac{1}{\sqrt{(a^2+u^2)^3}}(-au\sin v, au\cos v, a^2)$$

$$\vec{n}_{0_v}' = \frac{1}{\sqrt{(a^2+u^2)}}(a\cos v, a\sin v, 0) = \frac{a^2+u^2}{\sqrt{(a^2+u^2)^3}}(a\cos v, a\sin v, 0)$$

$$d\vec{n}_0 = \vec{n}_{0_u}' du + \vec{n}_{0_v}' dv =$$

$$= \frac{1}{(a^2+u^2)^{3/2}} \big(-au\sin v\,du + a(a^2+u^2)\cos v\,dv,$$
$$au\cos v\,du + a(a^2+u^2)\sin v\,dv,\ \ a^2du\big)$$

Dalje

$$d\vec{r} = \vec{r}\,'_u\,du + \vec{r}\,'_v\,dv =$$
$$= (\cos v\,du - u\sin v\,dv,\ \sin v\,du + u\cos v\,dv,\ a\,dv)$$

Sada, kako smo za diferencijalnu jednačinu linija krivina uzeli $d\vec{n}_0 = \lambda\, d\vec{r}$ imamo $\lambda = \frac{d\vec{n}_0}{d\vec{r}}$

$$\frac{-au\sin v\,du + a(a^2+u^2)\cos v\,dv}{\cos v\,du - u\sin v\,dv} = \frac{au\cos v\,du + a(a^2+u^2)\sin v\,dv}{\sin v\,du + u\cos v\,dv}$$

$$= \frac{a^2du}{a\,dv}$$

Izjednačavanjem bilo koja dva razlomka gornje jednačine dobijamo istu diferencijalnu jednačinu (npr. izjednačimo zadnja dva razlomka) $\underline{a^2u\cos v\,du\,dv} + a^2(a^2+u^2)\sin v\,dv^2$

$$= a^2\sin v\,du^2 + \underline{a^2u\cos v\,du\,dv} \quad /:a^2 \quad /:\sin v$$

$$(a^2+v^2)\,dv^2 = du^2$$

$$dv = \pm\frac{du}{\sqrt{a^2+u^2}}$$ čije je rešenje

$$v = \pm\ln(u+\sqrt{a^2+u^2}) + C$$

Linije krivine su

$$\vec{r}_{1,2} = (u\cos(\pm\ln(u+\sqrt{a^2+u^2})+C),\ u\sin(\pm\ln(u+\sqrt{a^2+u^2})+C),\ \pm a\ln(u+\sqrt{a^2+u^2})+C)$$

Zadaci za vježbu

na sljedećim površinama:

(1.) Odrediti linije krivine sljedećih površina

a) hiperboličkom paraboloidu: $z = axy$;

Uputa:

Diferencijalna linija krivine na površ $z = f(x,y)$ glasi

$$\begin{vmatrix} dy^2 & -dx\,dy & dx^2 \\ 1+q^2 & pq & 1+p^2 \\ t & s & r \end{vmatrix} = 0,$$

pa je $\ln\left(ay + \sqrt{1+a^2y^2}\right) \pm \ln\left(ax + \sqrt{1+a^2x^2}\right) = \text{const}.$

b) rotacionoj plohi

$x = f(u)\cos v,\ y = f(u)\sin v,\ z = \phi(u),\ u \in \mathbb{R},\ v \in [0, 2\pi]$;

Uputa:

Merdijani i paralele $(u = C_1,\ v = C_2)$.

c) kružnom valjku

$x = R\cos v,\ y = R\sin v,\ z = u,\ u \in \mathbb{R},\ v \in [0, 2\pi]$;

Uputa:

Koordinatne krive (paralele i merdijani, koji su izvodnice valjka).

d) kružnom stošcu (kupa)

$x = u\cos v,\ y = u\sin v,\ z = ku,\ u \in \mathbb{R},\ v \in [0, 2\pi]$;

Uputa:

Koordinatne krive (izvodnice stošca i njihove ortogonalne trajektorije).

e) na plohi $\vec{r} = (u^2+v^2,\ u^2-v^2,\ v),\ u, v \in \mathbb{R}$;

Uputa: koordinatne krive